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ON THE STABILIZATION OF STEADY-STATE MOTIONS OF MECHANICAL SYSTEMS*

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The effect is investigated of forces of specified structure on the stability of steady-state motions of nonholonomic (in particular, holonomic) mechanical systems. It is assumed /1,2/ that the forces are applied not just to the positional coordinates but also to the cyclic coordinates. In addition, the assumption is introduced that the controlling forces, applicable to the cyclic coordinates, depend on the positional velocities. Under their action in the reduced system there appear, in the general case, additional potential and gyroscopic forces /2/, and also dissipative-accelerative and nonconservative positional forces (such forces arise because of the nonholonomicity terms /3/). As a consequence of this there is obtained a possible stabilization in the first approximation of the unstable steady-state motions in the absence of a minimum of the potential energy and for an odd degree of instability. Here the characteristic equation of the first approximation has zero roots, but the system can be reduced to a special case /3,4/ by using the methods of Liapunov's theory of critical cases.

1. Let the position of a scleronomous nonholonomic system be determined by the generalized coordinates $q_1, q_2, \ldots, q_n$ and let the constraints imposed on the system have the form

$$q_\mu^{\cdot} = \beta_{\mu\rho}(q) q_\rho^{\cdot} \qquad (1.1)$$

Here and everywhere henceforth

$$\begin{gathered}\mu,\ \sigma = 1, 2, \ldots, m;\ \alpha,\ \beta,\ \nu,\ \delta = m+1,\ m+2, \ldots, m+k \\ \varkappa = 1, 2, \ldots, m-l;\ \ \xi, \eta = m-l+1, m-l+2, \ldots, m \\ i,\ j,\ r = m+k+1,\ \ m+k+2, \ldots, n;\ \ s,\ \rho = m+1, \\ m+2, \ldots, n\end{gathered}$$

Summation is carried out over twice repeated indices. As the variables characterizing the system's state we take the Routh variables $q_\sigma, q_\sigma^{\cdot}, q_i, q_i^{\cdot}, q_\alpha, p_\alpha$, where $p_\alpha = \partial T/\partial q_\alpha^{\cdot}$, $T = {}^1/_2 a_{\rho s}(q) q_\rho^{\cdot} q_s^{\cdot}$ is the system's kinetic energy expressed in terms of the independent velocities. We introduce into consideration the Routh function /5/

$$\begin{aligned}&R = L - p_\alpha q_\alpha^{\cdot} = R_2 + R_1 + R_0,\ L = T - \Pi(q) \\ &R_2 = {}^1/_2 a_{ij}^* q_i^{\cdot} q_j^{\cdot},\quad R_1 = \gamma_{\alpha i} q_i^{\cdot} p_\alpha \\ &R_0(q, p) = -{}^1/_2 b_{\alpha\beta} p_\alpha p_\beta - \Pi(q) \\ &\| b_{\alpha\beta} \| = \| a_{\alpha\beta} \|^{-1},\quad a_{ij}^* = a_{ij} - b_{\alpha\beta} a_{\alpha j} a_{\beta i},\quad \gamma_{\alpha i} = b_{\alpha\beta} a_{\beta i}\end{aligned}$$

($\Pi(q)$ is the potential energy). Then the equations of motion can be written as

$$q_\alpha^{\cdot} = -\frac{\partial R}{\partial p_\alpha},\qquad p_\alpha^{\cdot} = \frac{\partial R}{\partial q_\alpha} + B_{\mu\alpha}\frac{\partial R}{\partial q_\mu} + \left(\frac{\partial T_0}{\partial q_\mu^{\cdot}}\right) q_\rho^{\cdot} \Omega_{\mu\alpha\rho} + Q_\alpha \qquad (1.2)$$

$$\frac{d}{dt}\frac{\partial R}{\partial q_i^{\cdot}} - \frac{\partial R}{\partial q_i} - B_{\mu i}\frac{\partial R}{\partial q_\mu} = \left(\frac{\partial T_0}{\partial q_\mu^{\cdot}}\right) q_\rho^{\cdot} \Omega_{\mu i\rho} + Q_i \qquad (1.3)$$

$$\Omega_{\mu s\rho} = \frac{\partial B_{\mu s}}{\partial q_\rho} - \frac{\partial B_{\mu\rho}}{\partial q_s} + B_{\sigma\rho}\frac{\partial B_{\mu s}}{\partial q_\sigma} - B_{\sigma s}\frac{\partial B_{\mu\rho}}{\partial q_\sigma},\qquad \left(\frac{\partial T_0}{\partial q_\mu^{\cdot}}\right) = \theta_{\mu s} q_s^{\cdot}$$

Here T_0 is the expression of the system's kinetic energy neglecting the nonholonomic constraints (1.1), Q_s are the nonpotential generalized forces referred to the independent velocities.

We assume that q_α are coordinates cyclic in the sense of the definition in /6/. When

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$Q_s = 0$ let there exist a manifold of steady-state motions /7/, whose dimension is not less than the sum of the number of cyclic coordinates and the number of nonholonomic constraints of general form /3/. We assume that the first $m-l$ constraints (1.1) are of Chaplygin type, i.e., for corresponding coordinates $\partial(T_0 - \Pi)/\partial q_\varkappa = 0$, $\partial\beta_{\mu\rho}/\partial q_\varkappa = 0$. In addition, let the conditions

$$\theta_{\varkappa\beta}\Omega_{\varkappa\alpha\nu} = 0, \quad B_{\eta\alpha} = 0, \quad \frac{\partial R}{\partial q_\alpha} = 0, \quad \frac{\partial B_{\eta i}}{\partial q_\alpha} = 0$$

be fulfilled, under which the manifold mentioned is determined by the equations for the positional coordinates /3/. These equations, written in the Routh variables, are

$$\frac{\partial R_0}{\partial q_i} + B_{\eta i}\frac{\partial R_0}{\partial q_\eta} + \theta_{\varkappa\alpha}\Omega_{\varkappa i\nu}b_{\alpha\beta}b_{\nu\delta}p_\beta p_\delta = 0 \tag{1.4}$$

2. Let us consider the possibility of stabilizing the unstable steady-state motions by the application of forces of a specific structure to the positional and cyclic coordinates /1,2/. We take an arbitrary steady-state motion

$$q_\eta = q_{\eta 0}, \quad q_i = q_{i0}, \quad q_i^{\cdot} = 0, \quad p_\alpha = c_\alpha = \text{const} \tag{2.1}$$

from the manifold. Let the forces Q_s vanish on the steady-state motion (2.1), be independent of q_α and, in Routh variables, be expressed as

$$Q_i = -f_{i\beta}(q, q^{\cdot})p_\beta - f_{ij}(q, q^{\cdot})q_j^{\cdot} - p_{ij}q_j + F_i \tag{2.2}$$
$$Q_\alpha = f_{\alpha j}(q, q^{\cdot})q_j^{\cdot}, \quad p_{ij} = -p_{ji} = \text{const}$$

where F_i constant forces to be added on if necessary /2/ for the fulfillment of the equality $Q_{i0} = 0$ on the steady-state motion. Setting $q_i = q_{i0} + x_i$, $p_\alpha = c_\alpha + y_\alpha$, $q_\eta = q_{\eta 0} + s_\eta$, we set up the equations of perturbed motion for the equation of constraints of general form, for the second group of equations in (1.2) and for Eqs.(1.3). Having made linear approximations in them, we write them as

$$s^{\cdot} = Bx^{\cdot} + \Phi_1(x, s, x^{\cdot}) \tag{2.3}$$
$$y^{\cdot} = Nx^{\cdot} + F_2x^{\cdot} + \Phi_2(x, s, y, x^{\cdot})$$
$$Ax^{\cdot\cdot} + \Gamma y^{\cdot} + (D_1 + G_1)x^{\cdot} + (M + P)x + (H + F_1)y + Es = \Phi_3(x, s, y, x^{\cdot})$$

Here

$$A = \|(a_{ij}^*)_0\|, \quad B = \|(B_{\eta i})_0\|, \quad \Gamma = \|(\gamma_{\alpha i})_0\|$$
$$D_1 + G_1 = \|(g_{ri}^{\alpha})_0 c_\alpha + (\theta_{\varkappa\alpha}\Omega_{\varkappa i\delta}b_{\alpha\beta}\gamma_{\delta r} + \theta_{\varkappa\alpha}\Omega_{\varkappa i\nu}b_{\nu\beta}\gamma_{\alpha r})_0 c_\beta - (\theta_{\varkappa r}\Omega_{\varkappa i\delta} + \theta_{\mu\delta}\Omega_{\mu ir})_0(b_{\delta\beta})_0 c_\beta + (f_{ir})_0\|, \quad D_1 = D_1', \quad G_1 = -G_1'$$
$$N = \|(\theta_{\varkappa r}\Omega_{\varkappa\alpha\delta} + \theta_{\varkappa\delta}\Omega_{\varkappa\alpha r})_0(b_{\delta\beta})_0 c_\beta\|$$
$$F_2 = \|(f_{\alpha j})_0\|, \quad F_1 = \|(f_{i\alpha})_0\|, \quad P = \|p_{ir}\|$$
$$H = \left\|\left(\frac{\partial b_{\alpha\beta}}{\partial q_i} + B_{\eta i}\frac{\partial b_{\alpha\beta}}{\partial q_\eta}\right)_0 c_\beta - (\theta_{\varkappa\alpha}\Omega_{\varkappa i\nu}b_{\alpha\beta}b_{\nu\delta} + \theta_{\varkappa\alpha}\Omega_{\varkappa i\nu}b_{\delta\alpha}b_{\nu\delta})_0 c_\beta\right\|, \quad s' = \|s_1, s_2, \ldots, s_l\|$$
$$y' = \|y_1, y_2, \ldots, y_k\|, \quad x' = \|x_1, x_2, \ldots, x_{n-m-k}\|$$

$$E = \left\|\left\{\frac{\partial}{\partial q_\xi}\left[\frac{\partial R_0(q, c)}{\partial q_i} + B_{\eta i}\frac{\partial R_0(q, c)}{\partial q_\eta} + \theta_{\varkappa\alpha}\Omega_{\varkappa i\nu}b_{\alpha\beta}b_{\nu\delta}c_\beta c_\delta\right]\right\}_0\right\|$$
$$M = \left\|-\left\{\frac{\partial}{\partial q_j}\left[\frac{\partial R_0(q, c)}{\partial q_i} + B_{\eta i}\frac{\partial R_0(q, c)}{\partial q_\eta} + \theta_{\varkappa\alpha}\Omega_{\varkappa i\nu}b_{\alpha\beta}b_{\nu\delta}c_\beta c_\delta\right]\right\}_0\right\|$$
$$R_0(q, c) = -\frac{1}{2}b_{\alpha\beta}c_\alpha c_\beta - \Pi(q)$$

The zero index signifies that the corresponding expression is computed for the steady-state motion (2.1). The vector-valued functions Φ_1, Φ_2, Φ_3 contain nonlinear terms, and

$$\Phi_1(x, s, 0) = \Phi_2(x, s, y, 0) \equiv 0 \tag{2.4}$$

Note. In the case of cyclic coordinates such that the conditions /8/

$$B_{\sigma\alpha}=0,\ \frac{\partial B_{\sigma\rho}}{\partial q_\alpha}=0,\quad \frac{\partial(T_0-\Pi)}{\partial q_\alpha}=0$$

are fulfilled, the second group of equations in (1.2) take the form $p_\alpha^{\cdot}=Q_\alpha$. In this case $N=0$ in Eqs.(2.3), while in the equations for the positional coordinates in the conservative system (when $Q_s=0$) the dissipative-accelerative forces do not appear because of the nonholonomicity terms, in contrast to the more general case /3/.

In Eqs.(2.3) we make the linear substitution /9,10/

$$z=s-Bx,\ w=y-F_2x-Nx \tag{2.5}$$

Then the equations of perturbed motion become

$$z^{\cdot}=\Phi_1(x,\ z+Bx,\ x^{\cdot}),\ w^{\cdot}=\Phi_2(x,\ z+Bx,\ w+(F_2+N)x,x^{\cdot}) \tag{2.6}$$

$$Ax^{\cdot\cdot}+\Gamma w^{\cdot}+\Sigma_1x^{\cdot}+(H+F_1)w+\Sigma x+Ez=\Phi_3(x,z+Bx,w+(F_2+N)x,x^{\cdot}) \tag{2.7}$$

$$\Sigma_1=\Gamma(F_2+N)+G_1+D_1,\ \Sigma=(H+F_1)(F_2+N)+M+P+EB$$

The characteristic equation of the first approximation of this system is

$$\lambda^{k+l}\det\{A\lambda^2+\Sigma_1\lambda+\Sigma\}=0 \tag{2.8}$$

When $Q_s=0$ the characteristic equation turns into

$$\lambda^{k+l}\det\{A\lambda^2+(G_1+D_1+\Gamma N)\lambda+M+EB+HN\}=0 \tag{2.9}$$

If even one of the roots of Eqs.(2.8) or (2.9) lies in the right halfplane, then the steady-state motion (2.1) is unstable.

3. We present certain results on the stabilization of steady-state motions, analogous to the Thomson–Tate–Chetaev theorems.

Theorem 1. If the matrix $C=M+EB+HN$ is symmetric, is not positive-definite, and if $\det C\neq 0$, then of steady-state motion (2.1), under the action of forces with total dissipation with respect to positional velocities, remains unstable under the adding on of arbitrary positional-velocity-dependent gyroscopic forces to the positional coordinates.

Proof. Under the action of the forces mentioned the characteristic equation of the reduced system (2.7) is

$$\det\{A\lambda^2+(G_2+D_2)\lambda+C\}=0$$

where matrix D_2 is positive definite and $G_2=-G_2'$. By the theorem's conditions all the eigenvalues of matrix C are nonzero and among them there are negative ones. Then the theorem's assertion follows from the fourth Thomson–Tate–Chetaev theorem /12/.

Statement. If the symmetric part of matrix $M+EB$ is negative definite, then for $N=0$ and for an odd number of positional coordinates the steady-state motion cannot be stabilized by any generalized gyroscopic, dissipative-accelerative and nonconservative positional forces Q_i of form (2.2) applied to the positional coordinates.

The validity of this statement follows from Merkin's Theorem 9 /11/.

We now observe that the forces Q_α of form (2.2), applicable to the cyclic coordinates, and the nonholonomicity terms in the equations for the cyclic momenta are of like nature. Under their action, in general, in the reduced system (2.7) arise additional potential, nonconservative positional, gyroscopic and dissipative-accelerative forces, since the matrices $\Gamma(F_2+N)$, $H(F_2+N)$ decompose into symmetric and skew-symmetric ones.

Under the action of forces $Q_i=-F_1y$, depending on the cyclic momenta and applicable to the positional coordinates, for the reduced system there may arise as well additional potential and nonconservative forces of the form F_1Nx or F_1F_2x under the simultaneous action of these generalized forces and of the forces $Q_\alpha=F_2x^{\cdot}$. Moreover, the forces Q_i, depending on the cyclic momenta, do not yield dissipative-accelerative and gyroscopic forces for system (2.7). Such forces will not arise also under the action of forces $Q_\alpha=F_2x^{\cdot}$ if $\Gamma=0$ (or $R_1=0$). Thus, by applying a force to the cyclic coordinates we can in some cases stabilize unstable steady-state motions under the hypotheses of Theorem 1 and of the Statement, since under the action of these forces the potential energy of the reduced system can change. For stabilization we should choose the coefficients of forces Q_s in such a way that the roots of the reduced system's characteristic equation lie to the left of the imaginary axis, because in such case the next theorem (proved in /3/ with $Q_\alpha = 0$) is valid for steady-state motions defined by manifold (1.4).

Theorem 2. If all roots of characteristic Eq.(2.8), except $k+l$ zero ones, have negative real parts, then the singular case of $k+l$ zero roots obtains and the steady-state motion is asymptotically stable relative to the velocities $q_i^{\cdot}$ and is stable relative to the coordinates q_i, q_η and to the momenta p_α.

In Routh variables the proof of this theorem is simpler than the one in /3/. We write the Eqs.(2.7) of perturbed motion as

$$x^{\cdot} = x_1, \quad x_1^{\cdot} = -A^{-1}\{\Gamma y^{\cdot} + \Sigma_1 x_1 + (H + F_1) w + \Sigma x + Ez\} + \Phi_4(x, z, w, x^{\cdot}) \tag{3.1}$$

Since $\det \Sigma \neq 0$ under the theorem's conditions, a solution $u(z, w)$ exists of the equation

$$A^{-1}\Sigma u(z, w) + A^{-1}(H + F_1) w + A^{-1}Ez - \Phi_4^*(u, z, w, 0) = 0$$

where Φ_4^* is the part of the nonlinear terms Φ_4, containing the freely occurring critical variables z, w. In Eqs.(2.5) and (3.1) we make the change of variables $x = u(z, w) + \zeta$ /4,12/. Because the nonlinear terms Φ_1 and Φ_2 satisfy condition (2.4), these equations turn into a system where all the nonlinear terms vanish when $\zeta = x_1 = 0$, i.e., the singular case obtains /3, 4,12/. In such a case the steady-state motion (2.1) will be asymptotically stable relative to the variables ζ, x_1 and stable relative to z, w, and, in the original variables, will be asymptotically stable relative to the positional velocities and stable relative to the coordinates q_i, q_η and the momenta p_α.

Note. If Eqs.(2.7) do not contain freely occurring critical variables z, w, then the steady-state motion will be asymptotically stable relative to the positional velocities and coordinates and stable relative to q_η, p_α.

Example. Consider a disk on a rough horizontal plane /7/. The disk's position is determined by the coordinates x and y of the disk's contact with the plane and by the Euler angles θ, ψ, φ. The Lagrange function L_0, set up without taking constraints into account, and the equations of the nonholonomic constraints have the form

$$\begin{aligned} L_0 = {}^1\!/_2 m (x^{\cdot 2} + y^{\cdot 2}) + ma\,[y^{\cdot}(\theta^{\cdot}\cos\theta\cos\psi - \psi^{\cdot}\sin\theta\sin\psi) - \\ x^{\cdot}(\theta^{\cdot}\cos\theta\sin\psi + \psi^{\cdot}\sin\theta\cos\psi)] + {}^1\!/_2 (A + m/a^2)\,\theta^{\cdot 2} + {}^1\!/_2 (A\cos^2\theta + \\ ma^2\sin^2\theta)\,\psi^{\cdot 2} + {}^1\!/_2 C(\varphi^{\cdot} - \psi^{\cdot}\sin\theta)^2 - mga\cos\theta \end{aligned} \tag{3.2}$$

$$x^{\cdot} = a\varphi^{\cdot}\cos\psi, \quad y^{\cdot} = a\varphi^{\cdot}\sin\varphi \tag{3.3}$$

here m is the disk's mass, a is the radius, A is the equatorial moment of inertia, C is the polar moment of inertia. The system being examined is a Chaplygin system; here the expression for L, set up with due regard to constraints (3.3), is independent of coordinates φ, ψ

$$L = {}^1\!/_2\,[(A + ma^2)\,\theta^{\cdot 2} + (C + ma^2)(\varphi^{\cdot} - \psi^{\cdot}\sin\theta)^2 + A\psi^{\cdot 2}\cos^2\theta] - mga\cos\theta$$

We introduce the variables $p_1 = \partial L/\partial\varphi^{\cdot}$, $p_2 = \partial L/\partial\psi^{\cdot}$; then the Routh function is

$$\begin{aligned} R = {}^1\!/_2\,[(A + ma^2)\,\theta^{\cdot 2} - p_1^2 (C + ma^2)^{-1} - p_2^2 (A\cos^2\theta)^{-1} - \\ 2p_1p_2\sin\theta\,(A\cos^2\theta)^{-1} - p_1^2\mathrm{tg}^2\,\theta\,(A)^{-1}] - mga\cos\theta \end{aligned}$$

The manifold of steady-state motions is determined by the equation $\partial R_0/\partial\theta = 0$, since the nonholonomicity terms in the equation for coordinates θ vanish /7/ and the equation itself admits of the solution

$$\theta = \theta_0, \; p_1 = c_1 = \text{const}, \quad p_2 = c_2 = \text{const}$$

Let the scattering function have the form $F = {}^1\!/_2 h\theta^{\cdot 2}$ /7/. Setting $\theta = \theta_0 + \eta$, $p_1 = c_1 + y_1$, $p_2 = c_2 + y_2$, we write the equations of perturbed motion

$$\begin{aligned} & y_1^{\cdot} = \eta^{\cdot}\frac{ma^2}{A}\left[(c_1 + y_1)\,\mathrm{tg}\,(\theta_0 + \eta) + (c_2 + y_2)\frac{1}{\cos(\theta_0 + \eta)}\right] \\ & y_2^{\cdot} = -\eta^{\cdot} ma^2\cos(\theta_0 + \eta)[(c_1 + y_1)\left(\frac{1}{C + ma^2} + \frac{\mathrm{tg}^2(\theta_0 + \eta)}{A}\right) + \\ & (y_2 + c_2)\frac{\sin(\theta_0 + \eta)}{A\cos^2(\theta_0 + \eta)}] \\ & (A + ma^2)\,\eta^{\cdot\cdot} + h\eta^{\cdot} + \frac{2}{A}(y_1c_1 + y_2c_2)\frac{\mathrm{tg}\,\theta_0}{\cos^2\theta_0} + \\ & (c_1y_2 + c_2y_1)\frac{1 + \sin^2\theta_0}{A\cos^3\theta_0} - mga\eta\cos\theta_0 + \frac{1}{A}(c_1^2 + \\ & c_2^2)\left[\frac{\partial}{\partial\theta}\left(\frac{\mathrm{tg}\,\theta}{\cos^2\theta}\right)\right]_{\theta_0}\eta + \frac{c_1c_2}{A}\left[\frac{\partial}{\partial\theta}\left(\frac{1 + \sin^2\theta}{A\cos^3\theta}\right)\right]_{\theta_0}\eta + \cdots \end{aligned} \tag{3.4}$$

Let us study the stability of the disk's twisting about the vertical diameter $\theta = 0$, $p_1 = c_1 = 0$, $p_2 = c_2 = \text{const}$, $c_2 = \Omega A$, $\Omega = \psi_0^\cdot$ relative to the variables θ, p_1, p_2.

Having set up the first approximation equations and made a change of variables of form (2.5), we obtain the characteristic equation

$$(A + ma^2)\lambda^2 + h\lambda + (-mga + A\Omega^2 + ma^2\Omega^2) = 0$$

where the terms $ma^2\Omega^2$ arise because of the nonholonomicity terms in the first of Eqs.(3.4). The stability condition

$$\Omega^2 > \frac{mga}{A + ma^2} \tag{3.5}$$

coincides with the condition obtained in /7/. If now along the coordinate φ we add on the force $Q_\varphi = f\theta^\cdot$, then after a change of form (2.5) an additional potential force $f\Omega\eta$ appears in the reduced equation for θ and the stability condition becomes

$$-mga + \Omega^2(A + ma^2) + \Omega f > 0 \tag{3.6}$$

Consequently, having chosen $f > 0$, when $\Omega > 0$ and $f < 0$ when $\Omega < 0$ so as to fulfil condition (3.6), the unstable twisting can be stabilized when condition (3.5) is violated. We note that $R_1 = 0$, for the disk and, therefore, in the reduced system there do not appear additional dissipative-accelerative forces both because of the nonholonomicity terms in the first two of Eqs.(3.4) and under the action of forces Q_φ.

4. Obviously, analogous results will be valid for the steady-state motions of holonomic systems if the equations of perturbed motion are set up with due regard to perturbations of the cyclic momenta /2,5/. Here, for the application of the methods of the theory of critical cases, in the equations for the positional coordinates it is necessary to write out in detail greater than in /5/ all the terms containing the critical variables, namely, the perturbations of the cyclic momenta

$$a_{ij}^* x_j^{\cdot\cdot} + \gamma_{\alpha i} y_\alpha^\cdot + g_{ri}^\alpha (c_\alpha + y_\alpha) x_r^\cdot + \frac{\partial b_{\alpha\beta}}{\partial q_i} c_\alpha y_\beta - \frac{\partial R_0(q,c)}{\partial q_i} +$$
$$\left(\frac{\partial a_{ij}^*}{\partial q_r} - \frac{1}{2}\frac{\partial a_{rj}^*}{\partial q_j}\right) x_r^\cdot x_j^\cdot + \frac{1}{2}\frac{\partial b_{\alpha\beta}}{\partial q_i} y_\alpha y_\beta = Q_i - Q_{i0}$$
$$\left(g_{ri}^\alpha = \frac{\partial \gamma_{\alpha i}}{\partial q_r} - \frac{\partial \gamma_{\alpha r}}{\partial q_i}\right)$$

Under the action of forces Q_α of form (2.2), after the change $w = y - F_2 x$, we obtain the reduced system's characteristic equation

$$\lambda^k \det\{A\lambda^2 + (G_1 + D_1 + \Gamma F_2)\lambda + C_1 + P + (H + F_1)F_2\} = 0 \tag{4.1}$$
$$C_1 = \left\| -\left(\frac{\partial^2 R_0(q,c)}{\partial q_i \partial q_r}\right)_0 \right\|, \quad G_1 + D_1 = \|(g_{ri}^\alpha c_\alpha + f_{ri})_0\|$$
$$H = \left\| \left(\frac{\partial b_{\alpha\beta}}{\partial q_i}\right)_0 c_\beta \right\|$$

When $Q_\alpha = 0$ a theorem analogous to Theorem 2 leads to the result established in /5/. The possibility of stabilizing the steady-state motions of holonomic systems by forces applied to the cyclic coordinates has been proved by the method of Liapunov functions /2/, and additional potential and gyroscopic forces have been isolated in the reduced system. In the general case, under the action of forces Q_α of form (2.2), besides the forces mentioned there can also arise, as we see from Eq.(4.1), additional dissipative-accelerative and nonconservative positional forces.

Example. We consider a heavy solid body with one fixed point in the Kovalevskaia case, i.e., the principal moments of inertia are connected by the relation $A = B = 2C$, while the center of gravity is located on the principal inertia axis x. In this case the Lagrange function is /2/

$$L = {}^1/_2\,[2C(\psi^{\cdot 2}\sin^2\theta + \theta^{\cdot 2}) + C(\varphi^\cdot + \psi^\cdot\cos\theta)^2] - Px_0 \sin\theta\sin\varphi$$

where P is the body's weight, x_0 is the coordinate of the center of gravity. When $Q_s = 0$ we have the integral $p = \partial L/\partial\psi^\cdot = \text{const}$.

Let us consider the steady-state motion for which

$$p = c_1, \quad \varphi = \pi/2, \quad \theta = \theta_0 \tag{4.2}$$

where θ_0 is found from the equation

$$Px_0 - \frac{c_1^2}{C}\frac{\sin\theta_0}{(\sin^2\theta_0+1)^2} = 0$$

For this motion the coefficients of the reduced potential energy $W = -R_0(q, c)$ are

$$c_\varphi = -Px_0\sin\theta_0, \quad c_\theta = -\frac{c_1^2}{C}\frac{\cos^2\theta_0(1-3\sin^2\theta_0)}{(\sin^2\theta_0+1)^3}$$

Let us consider the case $c_\varphi > 0$; motion (4.2) is stable for the values $\sin^2\theta_0 > 1/3$, and unstable for $\sin^2\theta_0 < 1/3$. To the cyclic coordinate ψ we apply the force

$$Q_\psi = f\theta^\cdot \qquad (4.3)$$

Then for the reduced system, in the equation for θ an additional potential force appears and the coefficient c_θ becomes

$$c_\theta^* = c_\theta - \frac{c_1}{C}\frac{2\sin\theta_0\cos\theta_0}{(\sin^2\theta_0+1)^2}f$$

By choosing the magnitude and sign of coefficient f we can make c_θ^* positive when $\sin^2\theta_0 < 1/3$. Here the choice of the sign of f depends upon the sign of c_1. Under the action of force (4.3), for the reduced system an additional force

$$\frac{\cos\theta_0}{\sin^2\theta_0+1}f\theta^\cdot = d_3\theta^\cdot$$

appears in the equation for φ. In order to obtain the singular case we add on further the dissipative force $Q_\theta = -d_1\theta^\cdot, d_1 > 0$. Then the characteristic equation determining the nonzero roots of the reduced system is written as

$$a_{11}^*a_{22}^*\lambda^4 + a_{22}^*d_1\lambda^3 + (a_{22}^*c_\theta^* + a_{11}^*c_\varphi + g^2 - gd_3)\lambda^2 +$$
$$c_\varphi d_1\lambda + c_\varphi c_\theta^* = 0$$
$$g = c_1\sin\theta_0(2+\cos^2\theta_0)(\sin^2\theta_0+1)^{-2}$$

With the chosen magnitude and sign of f all the coefficients of the equation are postive (it can be verified that $-gd_3 > 0$). The Hurwitz criterion leads to the condition $g^2 - gd_3 > 0$, which too is fulfilled.

Thus, the motion unstable for $\sin^2\theta_0 < 1/3$ can be stabilized by applying force (4.3), wherein f is selected from the condition $c_\theta^* > 0$, and the dissipative force $Q_\theta = -d_1\theta^\cdot$ of arbitrary magnitude. If the dissipative force $Q_\varphi = -d_2\varphi^\cdot, d_2 > 0$ acts here, i.e., we have total dissipation with respect to the positional velocities, the stabilization achieved is not destroyed (the possibility of such a stabilization was noted in /2/).

The rotation of the Kovalevskaia top around the verical (with $\theta = \varphi = \pi/2$), in the unstable case when $x_0 > 0$ /13/, has been stablized in /2/ by another method. In this case the matrix H in (4.1) vanishes; therefore, additional potential forces HF_2x do not appear in the reduced system, but forces F_1F_2x can be obtained. Thus, if the forces $Q_\psi = f\varphi$, $Q_\varphi = -fy$ act on the body, then we obtain an additional potential force in the equation for φ. Then, under the action of forces with total dissipation along the positional velocities we can stabilize the rotation, unstable for $x_0 > 0$, aroung the vertical with the condition $f^2 > Px_0$, where the angular velocity $\omega = \psi^\cdot$ must satisfy the condition $\omega^2 > Px_0/C$, obtained in /13/.

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